\_<X>:=PolynomialRing(Rationals());

R<h,x>:=PolynomialRing(Rationals(),2);

S<h,x>:=FunctionField(Rationals(),2);

PS<t>:=PolynomialRing(S);

print "Z2xZ20 Case";

j10:=(h^6-4\*h^5+16\*h+16)^3\*(h+1)^(-2)\*(h-4)^(-1)\*h^(-5);

//j-invariant for 10-isogeny

E:=EllipticCurve([1,0,0,-36/(j10-1728),-1/(j10-1728)]);

WE:=WeierstrassModel(E);

//Indeed all such curves have a 2-torsion point over Q

D:=DivisionPolynomial(WE,2);

Dfact:=Factorization(D);

g:=x^3+aInvariants(WE)[2]\*x^2+aInvariants(WE)[4]\*x+aInvariants(WE)[5];

f:=Evaluate(g,[h,t+Roots(Dfact[1][1])[1][1]]);

//moving the 2-torsion point to (0,0)

f:=PS!f;

b:=Coefficients(f)[3];

d:=Coefficients(f)[2];

F:=(b^2-4\*d)\*d;

//want to know when F is a square, want y^2 = F, can absorb squares

numfact:=Factorization(Numerator(F));

denfact:=Factorization(Denominator(F));

FF:=R!1;

for i:=1 to #numfact do

 if IsOdd(numfact[i][2]) then

 FF:=FF\*R!numfact[i][1];

 end if;

end for;

for i:=1 to #denfact do

 if IsOdd(denfact[i][2]) then

 FF:=FF\*R!denfact[i][1];

 end if;

end for;

FF;

//FF:=h^3 - 3\*h^2 - 4\*h

E:=EllipticCurve(X^3-3\*X^2-4\*X);

Rank(E);

MordellWeilGroup(E);

PointSearch(E,1000);

//it turns out y^2 = FF is an elliptic curve with rank 0, torsion subgroup Z/2xZ/2,

//the points having x-coords: h=-1, h=4, h=0, all of which are cusps.

G:=d;

//want to know when G is a square, want y^2 = G, can absorb squares

numfact:=Factorization(Numerator(G));

denfact:=Factorization(Denominator(G));

GG:=R!1;

for i:=1 to #numfact do

 if IsOdd(numfact[i][2]) then

 GG:=GG\*R!numfact[i][1];

 end if;

end for;

for i:=1 to #denfact do

 if IsOdd(denfact[i][2]) then

 GG:=GG\*R!denfact[i][1];

 end if;

end for;

GG;

E:=EllipticCurve(X^3+X^2+4\*X+4);

Rank(E);

MordellWeilGroup(E);

PointSearch(E,1000);

//GG:=h^3+h^2+4\*h+4

//it turns out y^2 = GG is an elliptic curve with rank 0, torsion subgroup Z/6

//the points having x-coords: h=-1, h=4, h=0, all of which are cusps.

print "Z2xZ36 Case";

j18:=(h^3-2)^3\*(h^9-6\*h^6-12\*h^3-8)^3\*h^(-9)\*(h^3-8)^(-1)\*(h^3+1)^(-2);

//j-invariant for 18-isogeny

E:=EllipticCurve([1,0,0,-36/(j18-1728),-1/(j18-1728)]);

WE:=WeierstrassModel(E);

//Indeed all such curves have a 2-torsion point over Q

D:=DivisionPolynomial(WE,2);

Dfact:=Factorization(D);

g:=x^3+aInvariants(WE)[2]\*x^2+aInvariants(WE)[4]\*x+aInvariants(WE)[5];

f:=Evaluate(g,[h,t+Roots(Dfact[1][1])[1][1]]);

//moving the 2-torsion point to (0,0)

f:=PS!f;

b:=Coefficients(f)[3];

d:=Coefficients(f)[2];

F:=(b^2-4\*d)\*d;

//want to know when F is a square, want y^2 = F, can absorb squares

numfact:=Factorization(Numerator(F));

denfact:=Factorization(Denominator(F));

FF:=R!1;

for i:=1 to #numfact do

 if IsOdd(numfact[i][2]) then

 FF:=FF\*R!numfact[i][1];

 end if;

end for;

for i:=1 to #denfact do

 if IsOdd(denfact[i][2]) then

 FF:=FF\*R!denfact[i][1];

 end if;

end for;

FF;

//FF:=h^7 - 7\*h^4 - 8\*h

//Now we make the hyperelliptic curve y^2 = FF

PP<X,Y,Z>:=ProjectiveSpace(Rationals(),2);

C:=Curve(PP, Y^2\*Z^5 - X^7 + 7\*X^4\*Z^3 + 8\*X\*Z^6); C;

AutC:=Automorphisms(C);

G:=AutomorphismGroup(C); G; #G;

G.3;

GG:=AutomorphismGroup(C,[SchemeMap(G.3)]); GG;

Cg, prj:=CurveQuotient(GG); Cg; prj;

Degree(prj);

MordellWeilGroup(Cg);

//Shows there are only 4 points

//Need to pull back points to C

Points:=PointSearch(Cg,10000);

//Strangely it leaves out the points (1:1:1), we append it manually.

Append(~Points, Cg![1,1,1]);

Points;

for P in Points do

 preimageofP:= P @@ prj;

 RationalPoints(preimageofP);

end for;

prj(C![-1,0,1]);

prj(C![0,0,1]);

prj(C![2,0,1]);

prj(C![0,1,0]);

G:=d;

//want to know when G is a square, want y^2 = G, can absorb squares

numfact:=Factorization(Numerator(G));

denfact:=Factorization(Denominator(G));

GG:=R!1;

for i:=1 to #numfact do

 if IsOdd(numfact[i][2]) then

 GG:=GG\*R!numfact[i][1];

 end if;

end for;

for i:=1 to #denfact do

 if IsOdd(denfact[i][2]) then

 GG:=GG\*R!denfact[i][1];

 end if;

end for;

GG;

E:=EllipticCurve(X^3+1);

Rank(E);

MordellWeilGroup(E);

PointSearch(E,1000);

//GG:=h^3+1

//it turns out y^2 = GG is an elliptic curve with rank 0, torsion subgroup Z/6

//the points having x-coords: h=-1, h=2, h=0, all of which are cusps.