Math 2110Q - Multivariable Calculus
Name:

Section 15.9 Worksheet

1. Find the Jacobian of the transformations:

   (a) $x = \frac{1}{4}(u+v)$, $y = \frac{1}{4}(v-3u)$.

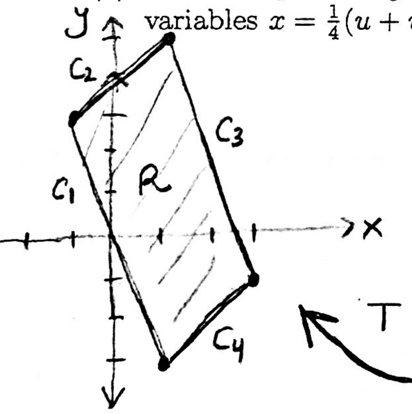
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{vmatrix} = \frac{1}{4}\cdot\frac{1}{4} - \frac{1}{4}\cdot\frac{-3}{4} = \frac{1}{16} + \frac{3}{16} = \boxed{\frac{1}{4}}$$

   (b) $x = \sqrt{2}u - \sqrt{2/3}v$, $y = \sqrt{2}u + \sqrt{2/3}v$.

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sqrt{2} & -\sqrt{2/3} \\ \sqrt{2} & \sqrt{2/3} \end{vmatrix} = \sqrt{2}\sqrt{2/3} + \sqrt{2}\sqrt{2/3} = 2\cdot\sqrt{2}\sqrt{2/3} = \boxed{4\cdot\sqrt{1/3}}$$

2. Find the image of the following sets under the given transformations, draw them out.

   (a) R = the parallelogram with vertices $(-1,3)$, $(1,-3)$, $(3,-1)$, and $(1,5)$ under the change of variables $x = \frac{1}{4}(u+v)$, $y = \frac{1}{4}(v-3u)$.

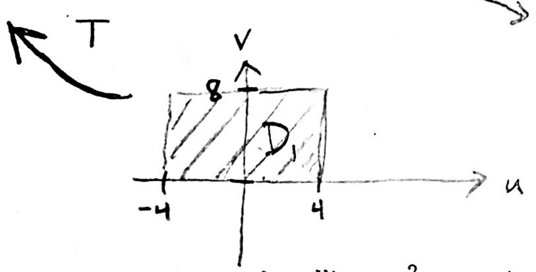


C_1: $y = -3x \longrightarrow \frac{1}{4}(v-3u) = -3\frac{1}{4}(u+v) \to v-3u = -3u-3v \to 4v=0 \to \boxed{v=0}$

C_2: $y = x+4 \longrightarrow \frac{1}{4}(v-3u) = \frac{1}{4}(u+v)+4 \to v-3u = u+v+16 \to -4u=16 \to \boxed{u=-4}$

C_3: $y = -3x+8 \longrightarrow \frac{1}{4}(v-3u) = -\frac{3}{4}(u+v)+8 \to v-3u = -3u-3v+32 \to 4v=32 \to \boxed{v=8}$

C_4: $y = x-4 \longrightarrow \frac{1}{4}(v-3u) = \frac{1}{4}(u+v)-4 \to \cancel{v}-3u = u+\cancel{v}-16 \to -4u=-16 \to \boxed{u=4}$

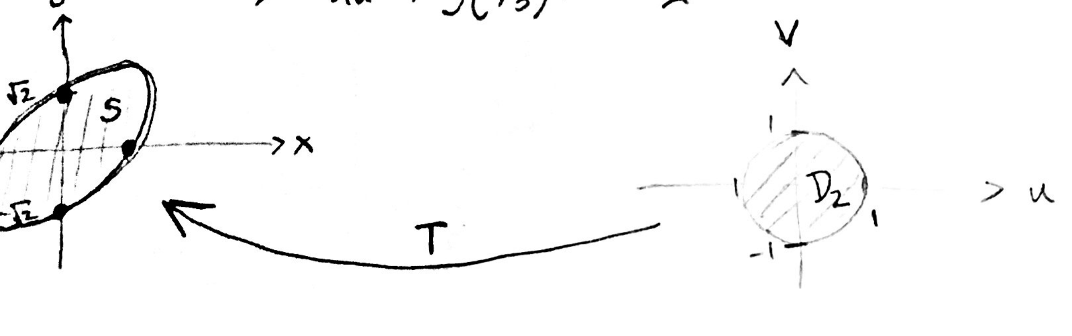


   (b) S = is the region bounded by the ellipse $x^2 - xy + y^2 = 2$ under the change of variables $x = \sqrt{2}u - \sqrt{2/3}v$, $y = \sqrt{2}u + \sqrt{2/3}v$. (Warning: don't muck it up, the answer is nice).

$$x^2 - xy + y^2 = 2 \longrightarrow (\sqrt{2}u - \sqrt{2/3}v)^2 - (\sqrt{2}u - \sqrt{2/3}v)(\sqrt{2}u + \sqrt{2/3}v) + (\sqrt{2}u + \sqrt{2/3}v)^2 = 2$$

$$\longrightarrow (2u^2 - 2\sqrt{2}\sqrt{2/3}uv + \tfrac{2}{3}v^2) - (2u^2 - \tfrac{2}{3}v^2) + (2u^2 + 2\sqrt{2}\sqrt{2/3}uv + \tfrac{2}{3}v^2) = 2$$

$$\longrightarrow 2u^2 + 3(\tfrac{2}{3})v^2 = 2 \longrightarrow u^2 + v^2 = 1$$



3. Calculate the following integrals using the transformations given:

(a) $\iint_R (4x+8y)\,dA$ where R is the region from 2.a, using the transformations from 1.a.

$$= \iint_{D_1} \left(4\left(\tfrac{1}{4}(u+v)\right)+8\left(\tfrac{1}{4}(u-3v)\right)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \int_0^8\int_{-4}^4 (u+v+2u-6v)\left(\tfrac{1}{4}\right) du\,dv$$

$$= \int_0^8\int_{-4}^4 \frac{3u+5v}{4}\,du\,dv = \frac{1}{4}\int_0^8 \frac{3u^2}{2}+5uv\Big|_{-4}^{4}\,dv$$

$$= \frac{1}{4}\int_0^8 (3\cdot 8+5\cdot 4v)-(3\cdot 8+5(-4)v)\,dv = \frac{1}{4}\int_0^8 40v\,dv = 10\left[\frac{v^2}{2}\right]_0^8$$

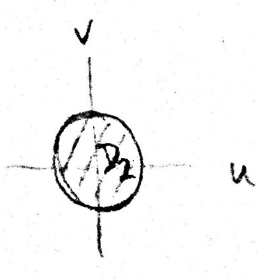
$$= 10\left[\frac{64}{2}\right] = \boxed{320}$$

(b) $\iint_S (x^2-xy+y^2)\,dA$ where S is the region from 2.b, using the transformations from 1.b. (Hint: You can be clever here and use a second change of coordinates to make this easier).

$$= \iint_{D_2}(u^2+v^2)\,dA = \iint_{D_2}(u^2+v^2)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv = \iint_{D_2}(u^2+v^2)\cdot\left(\frac{4}{\sqrt{3}}\right) du\,dv$$

$$= \frac{4}{\sqrt{3}}\iint_{D_2}(u^2+v^2)\,du\,dv$$

use $u = r\cos\theta$, $v = r\sin\theta$, $D_2 = \{(r,\theta) : 0\le r\le 1,\ 0\le\theta\le 2\pi\}$



$$= \frac{4}{\sqrt{3}}\int_0^{2\pi}\int_0^1 (r^2)\,r\,dr\,d\theta = \frac{4}{\sqrt{3}}\int_0^{2\pi} \frac{r^4}{4}\Big|_0^1 d\theta = \frac{4}{\sqrt{3}}\int_0^{2\pi}\frac{1}{4}\,d\theta$$

$$= \frac{4}{\sqrt{3}}\cdot\frac{1}{4}\cdot[2\pi-0] = \boxed{\frac{2}{\sqrt{3}}\pi}$$