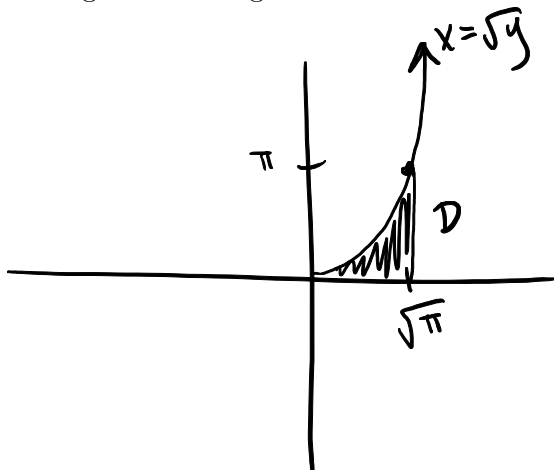


1. Evaluate $\int_0^\pi \int_{\sqrt{y}}^{\sqrt{\pi}} \frac{1}{2} y^{-\frac{1}{2}} \sin(9x^2) dx dy$ by reversing the order of integration using the following steps.

(a) Sketch the region this integral is defined over.

(5)



(b) Write the integral obtained by reversing the order of integration.

(5)

As Type I: $0 \leq y \leq x^2$
 $0 \leq x \leq \sqrt{\pi}$

$$\int_0^{\sqrt{\pi}} \int_0^{x^2} \frac{1}{2} y^{-\frac{1}{2}} \sin(9x^2) dy dx$$

(c) Evaluate the integral.

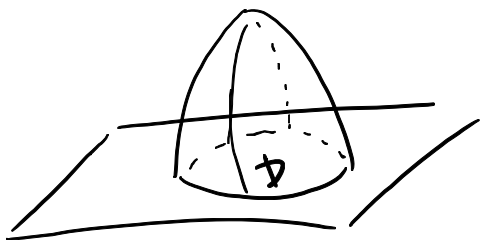
(5)

$$\int_0^{\sqrt{\pi}} \sin(9x^2) \left[y^{\frac{1}{2}} \right]_0^{x^2} dx = \int_0^{\sqrt{\pi}} x \sin(9x^2) dx \quad u = 9x^2$$

$$= \frac{1}{18} \left[-\cos(9x^2) \right]_0^{\sqrt{\pi}} = \frac{-1}{18} \left[\cos(9\pi) - \cos(0) \right] = \frac{-1}{18} (-2) = \frac{1}{9}$$

2. Use polar or cylindrical coordinates to set up an integral that is equal to the volume below the paraboloid $z = 18 - 2x^2 - 2y^2$ and above the xy -plane. [Do not evaluate the integral]

(9)



$$z=0 \quad \text{and} \quad z=18-2x^2-2y^2$$

$$0=18-2x^2-2y^2$$

↓

$$x^2+y^2=9 \quad \text{so}$$

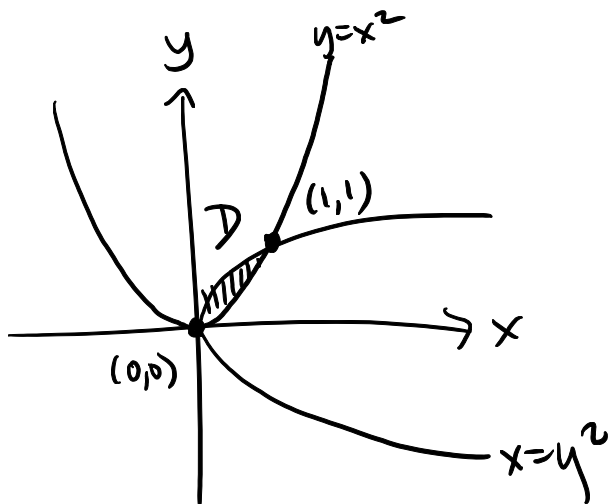
$$D = \{(r, \theta) : 0 \leq r \leq 3, \quad 0 \leq \theta \leq 2\pi\}$$

get:

$$\int_0^{2\pi} \int_0^3 (18-2r^2) r \, dr \, d\theta$$

3. Set up the following integrals but do not evaluate it.

- (a) Using rectangular coordinates, set up $\iiint_E x dV$ where E is bounded by the parabolic cylinders $y = x^2$ and $x = y^2$ and planes $z = 0$ and $z = x + y$. (9)

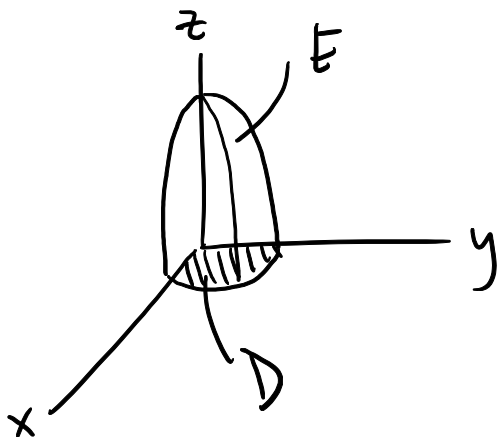


$$\iiint_E x dV = \iint_D \left[\int_0^{x+y} x dz \right] dA$$

as type I:

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} x dz dy dx$$

- (b) Using cylindrical coordinates, set up $\iiint_E (x + y + z) dV$ where E is the solid in the first octant that lies under the paraboloid $z = 4 - x^2 - y^2$. (9)



Triple type 1:

$$\int_0^{\pi/2} \int_0^2 \int_0^{4-r^2} (r \cos \theta + r \sin \theta + z) r dz dr d\theta$$

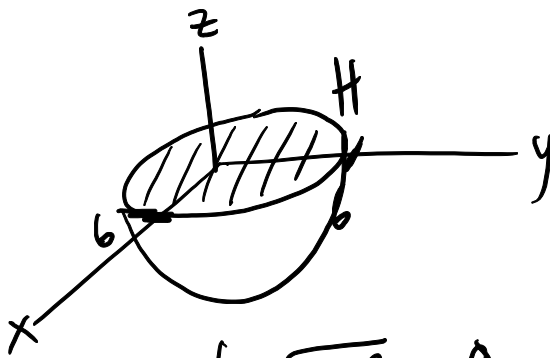
4. Consider

$$\iiint_H x(10 - x^2 - y^2 - z^2) dV,$$

where $H = \{(x, y, z) \mid x^2 + y^2 + z^2 \leq 36, z \leq 0\}$

(a) Set up but do not evaluate this integral in rectangular coordinates.

(9)



H:

$$-\sqrt{36-x^2-y^2} \leq z \leq 0$$

$$-\sqrt{36-x^2} \leq y \leq \sqrt{36-x^2}$$

$$-6 \leq x \leq 6$$

$$\int_{-6}^6 \int_{-\sqrt{36-x^2}}^{\sqrt{36-x^2}} \int_{-\sqrt{36-x^2-y^2}}^0 x(10-x^2-y^2-z^2) dz dy dx$$

(b) Set up but do not evaluate this integral in spherical coordinates.

(9)

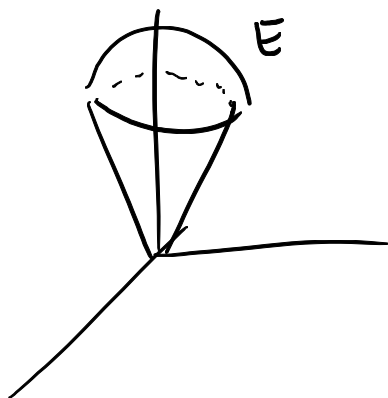
$$H: \quad 0 \leq \rho \leq 6, \quad 0 \leq \theta \leq 2\pi, \quad \frac{\pi}{2} \leq \phi \leq \pi$$

$$\int_{\pi/2}^{\pi} \int_0^{2\pi} \int_0^6 \rho \cdot \sin \phi \cos \theta (10 - \rho^2) \rho^2 \sin \phi d\rho d\theta d\phi$$

5. Using spherical coordinates, set up an integral that represents the volume of the solid that lies above the cone $\phi = \frac{\pi}{3}$ and below the sphere $\rho = 4 \cos \phi$. **Sketch this region.** [Do not evaluate the integral]

(8)

$$E: \{ (\rho, \theta, \phi) : \left. \begin{array}{l} 0 \leq \rho \leq 4 \cos \phi \\ 0 \leq \theta \leq 2\pi \\ 0 \leq \phi \leq \pi/3 \end{array} \right\}$$



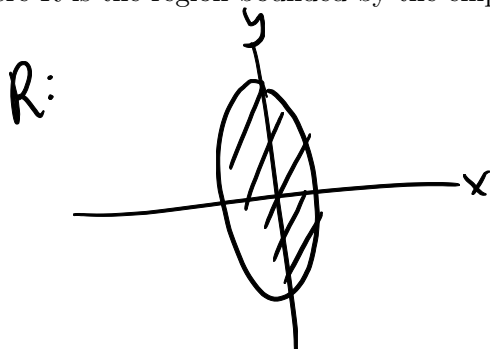
$$\int_0^{\pi/3} \int_0^{2\pi} \int_0^{4 \cos \phi} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi.$$

6. Use the transformation $x = 4u, y = 5v$ to **evaluate** the integral

(12)

$$\iint_R \frac{x^2}{16} + \frac{y^2}{25} dA$$

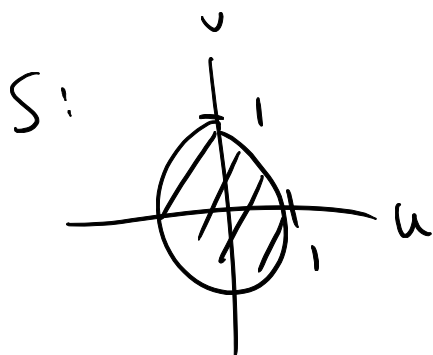
where R is the region bounded by the ellipse $25x^2 + 16y^2 = 400$.



$$S: 25(4u)^2 + 16(5v)^2 = 400$$

$$400u^2 + 400v^2 = 400$$

$$u^2 + v^2 = 1$$



$$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{vmatrix} 4 & 0 \\ 0 & 5 \end{vmatrix} = 20$$

$$\text{so } \iint_R \frac{x^2}{16} + \frac{y^2}{25} dA = \iint_S (u^2 + v^2)(20) du dv = 400 \iint_S (u^2 + v^2) du dv$$

$$\begin{aligned} u &= r \cos \theta \\ v &= r \sin \theta \end{aligned} \quad = 400 \int_0^{2\pi} \int_0^1 r^2 dr d\theta$$

$$= 400 (2\pi) \left(\frac{1}{3} \right)$$

7. Find the local maximum and minimum values and saddle points of the function

(10)

$$f(x, y) = xy - x^2y - xy^2.$$

Critical points:

$$0 = f_x = y - 2xy - y^2 \rightarrow \textcircled{1} y(1 - 2x - y) = 0$$

$$0 = f_y = x - x^2 - 2xy \rightarrow \textcircled{2} x(1 - x - 2y) = 0$$

① gives: $y = 0$ or $1 - 2x - y = 0$

↙ plug into ②

② $1 - 2x = y$
↓ plug into ②

$$x(1 - x) = 0$$

↙ $x = 0$ or $x = 1$
↓
 $(0, 0)$ $(1, 0)$

$$x(1 - x - 2(1 - 2x)) = 0$$

$x(-1 + 3x) = 0$
↓
 $x = 0$ or $x = \frac{1}{3}$

plugin @ ↓

plugin @ ↓

$$D = f_{xx}f_{yy} - f_{xy}^2 = 4xy - (1 - 2x - 2y)^2 \quad y = 1$$

$$y = \frac{1}{3}$$

↓
 $D(0, 0) = -1 < 0$ saddle

$D(1, 0) = -1 < 0$ saddle

$D(0, 1) = -1 < 0$ saddle

$D(\frac{1}{3}, \frac{1}{3}) = \frac{1}{3} > 0, \quad f_{xx}(\frac{1}{3}, \frac{1}{3}) = -\frac{2}{3} < 0$ local max.

8. Use Lagrange multipliers to find the extreme values of $f(x, y) = 8x^2 - 2y$ subject to the constraint $x^2 + y^2 = 1$.

(10)

$$\overbrace{g(x, y)}$$

$$\nabla f = \lambda \nabla g$$

$$\langle 16x, -2 \rangle = \lambda \langle 2x, 2y \rangle$$

System:

$$\textcircled{1} \quad 16x = \lambda 2x \longrightarrow \textcircled{1}: 16x - \lambda 2x = 0$$

$$\textcircled{2} \quad -2 = \lambda 2y \quad 2x(8 - \lambda) = 0$$

$$\textcircled{3} \quad x^2 + y^2 = 1$$

$$\downarrow$$

$$x = 0$$

or

$$\lambda = 8$$

← plug into $\textcircled{3}$

$$y^2 = 1 \text{ so } y = \pm 1$$

$$\downarrow$$

$$(0, 1), (0, -1)$$

$$f(0, 1) = -2 \leftarrow \text{abs. min}$$

$$f(0, -1) = 2$$

$$f\left(\sqrt{\frac{63}{64}}, -\frac{1}{8}\right) = \frac{65}{8} \leftarrow$$

abs. max

$$f\left(-\sqrt{\frac{63}{64}}, -\frac{1}{8}\right) = \frac{65}{8} \leftarrow$$

plug into $\textcircled{2} \downarrow$

$$-2 = 16y$$

$$y = -\frac{1}{8}$$

$$\downarrow$$

plug into $\textcircled{3}$

$$x^2 + \frac{1}{64} = 1$$

$$x = \pm \sqrt{\frac{63}{64}}$$

$$\downarrow$$

$$\left(\sqrt{\frac{63}{64}}, -\frac{1}{8}\right), \left(-\sqrt{\frac{63}{64}}, -\frac{1}{8}\right)$$

Bonus Problem 1:

Who's Theorem allows us to interchange the order of integration over rectangular domains?

Fubini's Theorem

Bonus Problem 2:¹

- Show via the Jacobian why for spherical coordinates $dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$.

See Book !

¹You must justify every step to receive credit.

Equations:

- $D = f_{xx}f_{yy} - f_{xy}^2$
- $m = \iint_D \rho(x, y) dA$
- Center of mass (\bar{x}, \bar{y}) given by:

$$\begin{aligned} - \bar{x} &= \frac{1}{m} \iint_D x \rho(x, y) dA \\ - \bar{y} &= \frac{1}{m} \iint_D y \rho(x, y) dA \end{aligned}$$

- Polar Coordinates:

$$\begin{aligned} - x &= r \cos \theta \\ - y &= r \sin \theta \\ - r^2 &= x^2 + y^2 \\ - \theta &= \arctan \frac{y}{x} \\ - dA &= r dr d\theta \end{aligned}$$

- Cylindrical Coordinates:

$$\begin{aligned} - x &= r \cos \theta \\ - y &= r \sin \theta \\ - z &= z \\ - r^2 &= x^2 + y^2 \\ - \theta &= \arctan \frac{y}{x} \\ - dV &= r dr d\theta dz \end{aligned}$$

- Spherical Coordinates:

$$\begin{aligned} - x &= \rho \sin \phi \cos \theta \\ - y &= \rho \sin \phi \sin \theta \\ - z &= \rho \cos \phi \\ - \rho^2 &= x^2 + y^2 + z^2 \\ - dV &= \rho^2 \sin \phi d\rho d\theta d\phi \end{aligned}$$

- Jacobian for two variables:

$$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$

- $\iint_R f(x, y) dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$

- Trigonometric Identities:

$$\begin{aligned} - \sin^2 \theta + \cos^2 \theta &= 1 \\ - \cos^2(x) &= \frac{1 + \cos(2x)}{2} \\ - \sin^2(x) &= \frac{1 - \cos(2x)}{2} \end{aligned}$$