Name:

Test 2 - Practice Questions

1. Give a definition of the following terms:
   (a) Vector space
   (b) Subspace
   (c) span$\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$
   (d) Nul$A$
   (e) Col$A$
   (f) Kernel$L$ (L a linear transformation)
   (g) Image$L$
   (h) Row$A$
   (i) dim V
   (j) basis of V
   (k) spanning set of V
   (l) rank A (for a matrix A)
   (m) nullity A
   (n) coordinate vector of $\vec{x}$ relative to a basis $\mathcal{B} = \{\vec{b_1}, \ldots \vec{b_n}\}$
   (o) Eigenvalue of A
   (p) Eigenvector of A
   (q) Eigenspace corresponding to λ
   (r) Characteristic polynomial of A
   (s) Multiplicity of an eigenvalue
   (t) Similar matrices
   (u) Diagonalizable

2. Give the definition of the following vector spaces. Include what $\vec{0}$ is in each space.
   (a) $\mathbb{P}_3$
   (b) $\mathbb{R}^5$
   (c) $M_{3,2}$
   (d) R^+

3. Determine which vector spaces each set is a sub*set* of. Then determine whether or not each subset is a subspace of that vector space.
   (a) $W = \left\{ \begin{bmatrix} a & 1 \\ b & -a \end{bmatrix} : a, b \in \mathbb{R} \right\}$

   (b) $H = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a & b & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$

   (c) $W = \left\{ \begin{bmatrix} 2x + y \\ -x \\ y - z \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$

   (d) $H = \{$ all polynomials of degree ≤ 2 of the form $a + bt^2$ for some $a, b \in \mathbb{R}\}$

   (e) $W = \{$ all polynomials of degree ≤ 3 with rational coefficients $\}$

   (f) $H = \text{Col}\, A$ where $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$

   (g) $W = \text{Nul}\, A$ with the same A given above

   (h) $H = \text{Row}\, A$ with the same A given above

   (i) $W = \{$ all even positive integers in $\mathbb{R}\}$ as a subset of $\mathbb{R}^+$.

   (j) $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \in \mathbb{R} \text{ and } x + y = 0 \right\}$

   (k) $W = \left\{ \begin{bmatrix} x + y \\ y - 1 \\ x + 2y \end{bmatrix} : x, y \in \mathbb{R} \right\}$

   (l) $H = \left\{ \begin{bmatrix} t \\ -t \\ t \\ -t \\ 2s \end{bmatrix} : t, s \in \mathbb{R} \right\}$

   (m) $W = \{$ all polynomials of degree ≤ 2 whose coefficients add up to 0 $\}$

   (n) $H = \{$ all polynomials of degree ≤ 2 whose coefficients add up to 1 $\}$

4. Determine a basis for each subspace in the previous question. Determine the dimension of each subspace.

5. For each of the given matrices, find a basis for Col$A$, Nul$A$, and Row$A$. Find the rank of A and nullity of A in each case.

(a) $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 2 & 1 & 0 & 0 \\ -1 & 0 & -2 & -2 \end{bmatrix}$

(d) $A = \begin{bmatrix} 2 & 1 \\ 0 & 0 \\ 1 & -1 \\ 2 & -4 \end{bmatrix}$

6. For each vector space V and basis $\mathcal{B}$ of V, determine the coordinate vector $[\vec{x}]_{\mathcal{B}}$ for the given vector $\vec{x}$.

(a) $V = \mathbb{R}^2$, $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$, $\vec{x} = \begin{bmatrix} 4 \\ 13 \end{bmatrix}$.

(b) $V = \mathbb{R}^3$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\}$, $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(c) $V = \mathbb{P}^1$, $\mathcal{B} = \{1 + t, 1 - t\}$, $\vec{x} = 3 + 2t$.

(d) $V = \mathbb{P}^2$, $\mathcal{B} = \{1, t - t^2, t\}$, $\vec{x} = 2 + t + t^2$.

(e) $V = M_{2,2}$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$, $\vec{x} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$.

7. If A is a 4×3 matrix, and rank$A$=2. What is the dimension of Nul$A$?

8. If A is a 2×6 matrix, what is the maximum rank of A? What is the minimum nullity of A?

9. Find the characteristic equation, eigenvalues, and eigenspaces corresponding to each eigenvalue of the following matrices:

$$\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

10. Which of the following vectors are eigenvectors of the matrix:

$$\begin{bmatrix} 1 & 3 & 6 \\ 2 & 1 & 4 \\ 1 & 0 & 3 \end{bmatrix}$$

(a) $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ (b) $\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$

11. Diagonalize the following matrices, if possible:

(a) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ $(\lambda = 1, -2)$ (c) $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$

12. For each matrix A that was diagonalizable from the previous question, find a formula for A^k. That is, find a single matrix whose entries are formulas in terms of k that determines A^k.

i.e.

$$\begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix}^k = \begin{bmatrix} -3 \cdot (-3)^k + 4 \cdot (-2)^k & 6 \cdot (-3)^k - 6 \cdot (-2)^k \\ -2 \cdot (-3)^k + 2 \cdot (-2)^k & 4 \cdot (-3)^k + -3 \cdot (-2)^k \end{bmatrix}$$

13. Find the eigenvalues of $\begin{bmatrix} 1 & k \\ 2 & 1 \end{bmatrix}$ in terms of k. Can you find an eigenvector corresponding to each of the eigenvalues?