Name:

Test 2 - Practice Questions

1. Give a definition of the following terms:
   (a) Vector space
   (b) Subspace
   (c) span$\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}\}$
   (d) Nul$A$
   (e) Col$A$
   (f) Kernel$L$ (L a linear transformation)
   (g) Image$L$
   (h) Row$A$
   (i) dim V
   (j) basis of V
   (k) spanning set of V
   (l) rank A (for a matrix A)
   (m) nullity A
   (n) coordinate vector of $\vec{x}$ relative to a basis $\mathcal{B} = \{\vec{b_1}, \ldots \vec{b_n}\}$
   (o) Eigenvalue of A
   (p) Eigenvector of A
   (q) Eigenspace corresponding to λ
   (r) Characteristic polynomial of A
   (s) Multiplicity of an eigenvalue
   (t) Similar matrices
   (u) Diagonalizable

   Solution: Look in the book's index to find each term

2. Give the definition of the following vector spaces. Include what $\vec{0}$ is in each space.
   (a) $\mathbb{P}_3$

   Solution: All polynomials of degree ≤ 3. The zero vector is the polynomial: 0.

   (b) $\mathbb{R}^5$

   Solution: Column vectors with 5 entries from the real numbers. $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

   (c) $M_{3,2}$

Solution: All 3 by 2 matrices with entries in the real numbers. $\vec{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

(d) R^+

Solution: All positive real numbers where $a \oplus b = a \cdot b$ and $c * a = a^c$. The zero vectors is: $\vec{0} = 1$.

3. Determine which vector spaces each set is a sub*set* of. Then determine whether or not each subset is a subspace of that vector space.

(a) $W = \left\{ \begin{bmatrix} a & 1 \\ b & -a \end{bmatrix} : a, b \in \mathbb{R} \right\}$

Solution: $W \subseteq M_{2,2}$, not a subspace since $\vec{0} \notin W$.

(b) $H = \left\{ \begin{bmatrix} 0 & 0 & 0 \\ a & b & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$

Solution: $H \subseteq M_{2,3}$. Yes it is a subspace. A basis is $\left\{ \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$

(c) $W = \left\{ \begin{bmatrix} 2x + y \\ -x \\ y - z \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$

Solution: $W \subseteq \mathbb{R}^3$. Yes it is a subspace. A basis is $\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$

(d) $H = \{$ all polynomials of degree ≤ 2 of the form $a + bt^2$ for some $a, b \in \mathbb{R}\}$

Solution: $H \subseteq \mathbb{P}_2$. Yes it is a subspace. A basis is $\{1, t^2\}$.

(e) $W = \{$ all polynomials of degree ≤ 3 with rational coefficients $\}$

Solution: $W \subseteq \mathbb{P}_3$. No it is not a subspace, it is not closed under scalar multiplication.

(f) $H = \text{Col}\, A$ where $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \\ 0 & 1 \end{bmatrix}$

Solution: $H \subseteq \mathbb{R}^3$. Yes it is a subspace

(g) $W = \text{Nul}\, A$ with the same A given above

Solution: $W \subseteq \mathbb{R}^2$. Yes it is a subspace

(h) $H = \text{Row}\, A$ with the same A given above

Solution: $H \subseteq \mathbb{R}^2$. Yes it is a subspace

(i) $W = \{$ all even positive integers in $\mathbb{R}\}$ as a subset of $\mathbb{R}^+$.

Solution: $W \subseteq \mathbb{R}^+$. No it is not a subspace, $\vec{0} \notin W$.

(j) $H = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \in \mathbb{R} \text{ and } x + y = 0 \right\}$

Solution: $H \subseteq \mathbb{R}^2$. Yes it is a subspace. A basis is $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.

(k) $W = \left\{ \begin{bmatrix} x+y \\ y-1 \\ x+2y \end{bmatrix} : x, y \in \mathbb{R} \right\}$

Solution: $W \subseteq \mathbb{R}^3$. No it is not a subspace since $\vec{0} \notin W$.

(l) $H = \left\{ \begin{bmatrix} t \\ -t \\ t \\ -t \\ 2s \end{bmatrix} : t, s \in \mathbb{R} \right\}$

Solution: $H \subseteq \mathbb{R}^5$. Yes it is a subspace.

(m) $W = \{$ all polynomials of degree ≤ 2 whose coefficients add up to 0 $\}$

Solution: $W \subseteq \mathbb{P}_2$. Yes it is a subspace. A basis is $\{1-t, 1-t^2\}$.

(n) $H = \{$ all polynomials of degree ≤ 2 whose coefficients add up to 1 $\}$

Solution: $H \subseteq \mathbb{P}_2$. No it is not a subspace. No zero vector.

4. Determine a basis for each subspace in the previous question. Determine the dimension of each subspace.

5. For each of the given matrices, find a basis for Col$A$, Nul$A$, and Row$A$. Find the rank of A and nullity of A in each case.

(a) $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

Solution:

Col A = span $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \right\}$. rank A = 2, nullity $A = 1$.

(b) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$

Solution:

Row A = span $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

rank $A = 3$, nullity $A = 0$. Nul $A = \left\{ \vec{0} \right\}$.

(c) $A = \begin{bmatrix} 1 & -2 & 0 & 3 \\ 2 & 1 & 0 & 0 \\ -1 & 0 & -2 & -2 \end{bmatrix}$

(d) $A = \begin{bmatrix} 2 & 1 \\ 0 & 0 \\ 1 & -1 \\ 2 & -4 \end{bmatrix}$

Solution: Nul A = $\left\{ \vec{0} \right\}$. nullity A = 0, rank $A = 2$.

6. For each vector space V and basis $\mathcal{B}$ of V, determine the coordinate vector $[\vec{x}]_{\mathcal{B}}$ for the given vector $\vec{x}$.

(a) $V = \mathbb{R}^2$, $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$, $\vec{x} = \begin{bmatrix} 4 \\ 13 \end{bmatrix}$.

Solution: $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 17 \\ 30 \end{bmatrix}$.

(b) $V = \mathbb{R}^3$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \right\}$, $\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(c) $V = \mathbb{P}^1$, $\mathcal{B} = \{1+t, 1-t\}$, $\vec{x} = 3 + 2t$.

Solution: $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 5/2 \\ 1/2 \end{bmatrix}$.

(d) $V = \mathbb{P}^2$, $\mathcal{B} = \{1, t - t^2, t\}$, $\vec{x} = 2 + t + t^2$.

Solution: $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

(e) $V = M_{2,2}$, $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$, $\vec{x} = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$.

7. If A is a 4×3 matrix, and rank$A$=2. What is the dimension of Nul$A$?

Solution: dim Nul A = nullity $A = 1$

8. If A is a 2×6 matrix, what is the maximum rank of A? What is the minimum nullity of A?

Solution: rank $A \leq 2$, nullity $A \geq 4$.

9. Find the characteristic equation, eigenvalues, and eigenspaces corresponding to each eigenvalue of the following matrices:

$$\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solution:

$\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$, $\lambda = 5, -2$, with corresponding eigenvectors $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$. (The eigenspaces are the span of these eigenvectors).

$\begin{bmatrix} 5 & 3 \\ -4 & 4 \end{bmatrix}$, SORRY THIS ONE HAS COMPLEX EIGENVALUES WHICH WE DID NOT LEARN.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$, $\lambda_1 = 1, \lambda_2 = 0$, with corresponding eigenspaces $W_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ and $W_2 = \text{span}\left\{ \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

$\begin{bmatrix} 2 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$, $\lambda_1 = 2, \lambda_2 = 3$, with corresponding eigenspaces $W_1 = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ and $W_2 = \text{span}\left\{ \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \right\}$.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\lambda = 1$, with eigenspace all of $\mathbb{R}^3$ (this is the identity matrix, so it times any vector is the same as that vector).

10. Which of the following vectors are eigenvectors of the matrix:

$$\begin{bmatrix} 1 & 3 & 6 \\ 2 & 1 & 4 \\ 1 & 0 & 3 \end{bmatrix}$$

(a) $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ (b) $\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$

Solution: Just check if $A\vec{x} = \lambda\vec{x}$ for some scalar λ. It turns out only $\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$ is an eigenvector, with eigenvalue 1.

11. Diagonalize the following matrices, if possible:

(a) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ $(\lambda = 1, -2)$ (c) $\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$

Solution: (a) is not diagonalizable, the only eigenvalue of the matrix is 2, and the eigenspace corresponding to $\lambda = 2$ is span$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$, and so there are not 2 linearly independent eigenvectors of this matrix. Therefore there is not a basis for $\mathbb{R}^2$ made of eigenvectors of this matrix, so it is not diagonalizable.

(b) One diagonalization is as follows:

$$\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} = PDP^{-1}$$

where $P = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(c) is not diagonalizable. Similar to part (a), it is impossible to find a basis for $\mathbb{R}^4$ of eigenvectors of this matrix.

12. For each matrix A that was diagonalizable from the previous question, find a formula for A^k. That is, find a single matrix whose entries are formulas in terms of k that determines A^k.

i.e.

$$\begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix}^k = \begin{bmatrix} -3 \cdot (-3)^k + 4 \cdot (-2)^k & 6 \cdot (-3)^k - 6 \cdot (-2)^k \\ -2 \cdot (-3)^k + 2 \cdot (-2)^k & 4 \cdot (-3)^k + -3 \cdot (-2)^k \end{bmatrix}$$

Solution: So we only need to do this for (b). Using the diagonalization we found:

$$\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix} = PDP^{-1}$$

where $P = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

From our work in class see that

$$\begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}^k = PD^kP^{-1} = P \begin{bmatrix} (-2)^k & 0 & 0 \\ 0 & (-2)^k & 0 \\ 0 & 0 & 1^k \end{bmatrix} P^{-1}$$

Now computing the product we get:

$$P \begin{bmatrix} (-2)^k & 0 & 0 \\ 0 & (-2)^k & 0 \\ 0 & 0 & 1^k \end{bmatrix} P^{-1} = \begin{bmatrix} -(-2)^k & -(-2)^k & 1 \\ 0 & (-2)^k & -1 \\ -(-2)^k & 0 & 1 \end{bmatrix} P^{-1} = \begin{bmatrix} (-2)^k & (-2)^k & 0 \\ 0 & (-1)^k(2)^{k+1} & -1 \\ (-2)^k & 0 & 1 \end{bmatrix}$$

13. Find the eigenvalues of $\begin{bmatrix} 1 & k \\ 2 & 1 \end{bmatrix}$ in terms of k. Can you find an eigenvector corresponding to each of the eigenvalues?

Solution: Eigenvalues: $\lambda_1 = 1 - \sqrt{2k}$, $\lambda_2 = 1 + \sqrt{2k}$.
Corresponding eigenvectors: $\vec{v}_1 = \begin{bmatrix} -\sqrt{k/2} \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} \sqrt{k/2} \\ 1 \end{bmatrix}$.