Name:

Test 1 - Practice Questions

1. Which of the following matrices are in row echelon form? Which are in reduced row echelon form?

$$\begin{bmatrix}1&3&5\\2&3&0\\1&0&0\end{bmatrix},\begin{bmatrix}1&0&0&0\\0&1&2&0\\0&0&0&1\end{bmatrix},\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix},\begin{bmatrix}1&0&2&3\\0&1&0&1\\0&1&2&0\end{bmatrix},\begin{bmatrix}1&1&0&1\\0&0&1&0\\0&0&0&1\end{bmatrix},\begin{bmatrix}0&0&0&0\\1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}$$

Solution: The 2nd, 3rd, and 5th are in row echelon form. The 2nd is the only one in reduced row echelon form.

2. Solve the following system of equations:

$$\begin{array}{rcrcrcr} & & x_2 & + & 5x_3 & = & -4\\ x_1 & + & 4x_2 & + & 3x_3 & = & -2\\ 2x_1 & + & 7x_2 & + & x_3 & = & -2\end{array}$$

Solution: Putting the coefficients into a matrix we obtain the augmented matrix:

$$\begin{bmatrix}0&1&5&-4\\1&4&3&-2\\2&7&1&-2\end{bmatrix}$$

Now we put this matrix into reduced row echelon form and obtain:

$$\begin{bmatrix}1&0&-17&0\\0&1&5&0\\0&0&0&1\end{bmatrix}$$

So since the last row gives the equation $0 = 1$, this system is inconsistent.

3. Solve the following system of equations:

$$\begin{array}{rcrcrcr} 2x_1 & & & - & 6x_3 & = & -8\\ & & x_2 & + & 2x_3 & = & 3\\ 3x_1 & + & 6x_2 & - & 2x_3 & = & -4\end{array}$$

Solution: Putting the coefficients into a matrix we obtain the augmented matrix:

$$\begin{bmatrix}2&0&-6&-8\\0&1&2&3\\3&6&-2&-4\end{bmatrix}$$

Now we put this matrix into reduced row echelon form and obtain:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

So we obtain the solutions $x_1 = 2, x_2 = -1, x_3 = 2$.

4. (a) Is $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ in span$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \right\}$? What about $\begin{bmatrix} \pi \\ \log_2 3 \\ 17 \end{bmatrix}$?

Solution: We can form the matrix whose columns are our vectors:

$$\begin{bmatrix} 1 & 3 & 0 \\ 2 & 4 & 2 \\ 0 & 3 & 3 \end{bmatrix}$$

and put this matrix into rref:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and since there is a pivot in each row, (i.e. no row of zeros), the vectors span $\mathbb{R}^3$, so both vectors must be in the span.

(b) Is $\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$? Is $\begin{bmatrix} \pi \\ \log_2 3 \\ 17 \end{bmatrix}$?

Solution: By the definition of span, these vectors must be linear combinations of those three vectors.

5. Let

$$A = \begin{bmatrix} 1 & 4 & -1 \\ 1 & 5 & 0 \\ 0 & 3 & 3 \end{bmatrix}.$$

(a) Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in the span of the columns of A? What about $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$?

Solution: If we put A into RREF, we see that there actually is a row of zeros, so we must check these vectors individually. First let's check $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Create the augmented matrix:

$$\begin{bmatrix} 1 & 4 & -1 & 1 \\ 1 & 5 & 0 & 2 \\ 0 & 3 & 3 & 3 \end{bmatrix}$$

Then put it into RREF to see if there is a solution to this system of equations: We obtain:

$$\begin{bmatrix} 1 & 0 & -5 & -3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So this system is consistent, so $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ **IS** in the span.

Now let's check $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Create the augmented matrix:

$$\begin{bmatrix} 1 & 4 & -1 & 3 \\ 1 & 5 & 0 & 2 \\ 0 & 3 & 3 & 1 \end{bmatrix}$$

Then put it into RREF to see if there is a solution to this system of equations:

$$\begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The last row gives the equation 0=1, so this system is inconsistent. Thus, $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is **NOT** in the span.

(b) Is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ a linear combination of the columns of A? What about $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$?

Solution: Similar to the previous question, by the definition of span, if a vector is in the span of the columns of A if and only if is a linear combination of the columns of A. Thus, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ **IS** a linear combination of the columns of A, and $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ is **NOT** a linear combination of the columns of A

6. Suppose $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(a) Give an example of a vector in span S but not in S.

Solution: Any linear combination of vectors in S is in span S. So for instance we can take $2 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$.

(b) Give an example of a vector **NOT** in span S.

Solution: If a vector $\vec{v}$ is in span S, then

$$\vec{v} = c \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ 2c \\ 0 \\ 3c \end{bmatrix} + \begin{bmatrix} 0 \\ d \\ d \\ 0 \end{bmatrix} = \begin{bmatrix} c \\ 2c + d \\ d \\ 3c \end{bmatrix}$$

In particular, notice the 4th entry must be 3 times the 1st entry. So to get a vector not in the span of S, just give an example of a vector in $\mathbb{R}^4$ whose 4th entry is NOT 3 times its 1st entry. For example:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

7. Find a vector $\vec{x}$ such that

$$\begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \\ 6 & 2 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$$

Solution: This is a matrix equation. To find the solutions, simply solve the augmented matrix:

$$\begin{bmatrix} 2 & 4 & 6 & 2 \\ 4 & 6 & 2 & 6 \\ 6 & 2 & 4 & 4 \end{bmatrix}$$

Putting it into RREF we obtain:

$$\begin{bmatrix} 1 & 0 & 0 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & -\frac{1}{3} \end{bmatrix}$$

yielding solutions $x_1 = \frac{2}{3}, x_2 = \frac{2}{3}, x_3 = -\frac{1}{3}$. So our vector $\vec{x}$ should be

$$\vec{x} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}$$

8. Calculate the following matrix products if they are defined, otherwise state they are undefined.

(a) $\begin{bmatrix} 1 & 0 & 2 \\ 1 & -1 & 1 \\ 0 & -1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Solution: $\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

Solution: $\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$

Solution: $\begin{bmatrix} 1 & 3 \\ 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 0 \\ 4 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 0 & 0 \end{bmatrix}$

Solution: $\begin{bmatrix} 4 & -1 \\ 8 & 4 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$

Solution: Undefined, 2x3 times a 2x2

(f) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 & 5 \\ 7 & 13 & 4 \\ -2 & 15 & -17 \end{bmatrix}$

Solution: $\begin{bmatrix} 2 & 4 & 5 \\ 7 & 13 & 4 \\ -2 & 15 & -17 \end{bmatrix}$ note that the identity matrix times any matrix always gives back the same matrix.

9. (a) Write $\begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$ as a linear combination of the vectors $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$?

Solution: We wish to solve

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$$

This is a vector equation which we solve by making the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 4 \end{bmatrix}$$

and solving it. I leave that part to you. (Put into RREF)

(b) Is the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ linearly independent?

Solution: We need to check if there are any nontrivial solutions to:

$$c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We check this by making the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

and seeing if there is a free variable. I leave that part to you. (Put into RREF, see if one column pertaining to a variable does not have a pivot). The answer is that there are no free variables, so the set is linearly independent.

(c) Do these vectors span $\mathbb{R}^3$?

We have a theorem that helps us with this. We form the matrix

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

and check whether there is a pivot in each row (when in REF), i.e. that there are no rows of zeros. If there are no rows of zeros, then by a theorem we have discussed in class, the columns of this matrix span $\mathbb{R}^3$. Here, the columns of our matrix are exactly the vectors.
The solution is YES they do span $\mathbb{R}^3$.

10. Determine whether the following sets are linearly independent:

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

(b) $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$

(d) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

(e) $\left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$

(f) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

Solution: The idea is to check if $c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n} = \vec{0}$ has any non-trivial solutions just like in the problem before. Key things to remember here are that

- if a set contains the zero vector, then the set is linearly dependent
- if a set contains more vectors than the dimension of the vectors (# of entries), then the set is linearly dependent

The answers are: yes, yes, no, no, no, no.

11. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the transformation defined by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} x+z \\ y+z \end{bmatrix}$$

(a) Show that T is a linear transformation.

Solution: We must check the two properties that define a linear transformation:

- For any $\vec{u}, \vec{v}$, $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$.
- For any $\vec{u}$, c, $T(c\vec{u}) = cT(\vec{u})$.

Let us define arbitrary vectors

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

Now simply compute both sides of each equation.

$$T(\vec{u}+\vec{v}) = T\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}\right) = \begin{bmatrix} (u_1+v_1)+(u_3+v_3) \\ (u_2+v_2)+(u_3+v_3) \end{bmatrix}$$

$$T(\vec{u}) + T(\vec{v}) = \begin{bmatrix} u_1+u_3 \\ u_2+u_3 \end{bmatrix} + \begin{bmatrix} v_1+v_3 \\ v_2+v_3 \end{bmatrix} = \begin{bmatrix} u_1+u_3+v_1+v_3 \\ u_2+u_3+v_2+v_3 \end{bmatrix}$$

and by rearranging we see that $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$.
Do the same to check $T(c\vec{u}) = cT(\vec{u})$.

(b) Determine the standard matrix for T.

Solution: To find the standard matrix for T, we must find were T sends the standard basis of the domain of T, in this case $\mathbb{R}^3$.
So, we will calculate:

$$T(\vec{e_1}) = T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T(\vec{e_2}) = T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$T(\vec{e_3}) = T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

And now we form the matrix by concatenating these vectors:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

and this matrix A is the standard matrix for T. We can double check that

$$A\vec{x} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+z \\ y+z \end{bmatrix}$$

(c) Is T onto?

Solution: There is a theorem which tells you that T is onto if and only if the columns of the standard matrix of T, that is the matrix A we just found, span the range of T, in this case $\mathbb{R}^2$. So we need to check is the columns of

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

span $\mathbb{R}^2$. We have a theorem that says that the columns of a matrix span $\mathbb{R}^n$ precisely when there no row of zeros in RREF, (there is a pivot in every row). So we put A into RREF, which it conveniently already is in, and notice that A has no row of zeros, (it has a pivot in every row). Therefore, the columns of A span $\mathbb{R}^2$, and therefore T is onto.

(d) Is T one-to-one?

Solution: There is a theorem which tells you that T is one-to-one if and only if the columns of the standard matrix of T, that is the matrix A we just found, are linearly independent. So we must check if the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

is linearly independent.

For more detailed steps, see solutions to previous problems on showing sets of vectors are linearly independent.

We form the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

and put it into RREF. Conveniently it already is in RREF, and we see that c_3 is a free variable, and thus this set of vectors is not linearly independent, the set is linearly dependent. Thus, T is not one-to-one.

12. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 \\ x_1 - x_2 \end{bmatrix}$$

(a) Show that T is a linear transformation.

Solution: See previous problem for idea.

(b) Determine the standard matrix for T.

Solution: See previous problem for idea, the answer is

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

(c) Is T onto?

Solution: See previous problem for idea, the answer is yes.

(d) Is T one-to-one?

Solution: See previous problem for idea, the answer is yes.

13. Compute the determinant and calculate the inverses of the following matrices:

(a)

$$\begin{bmatrix} 3 & 2 \\ 8 & 5 \end{bmatrix}$$

Solution:
determinant: -1 inverse: $\begin{bmatrix} -5 & 2 \\ 8 & -3 \end{bmatrix}$

(b)

$$\begin{bmatrix} 7 & 3 \\ -6 & -3 \end{bmatrix}$$

Solution:
determinant: -3 inverse: $\begin{bmatrix} 1 & 1 \\ -2 & -7/3 \end{bmatrix}$

(c)

$$\begin{bmatrix} 2 & -4 \\ 4 & -6 \end{bmatrix}$$

Solution:
determinant: 4 inverse: $\begin{bmatrix} -3/2 & 1 \\ -1 & 1/2 \end{bmatrix}$

(d)

$$\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$$

Solution:
determinant: 2 inverse: $\begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}$

(e)

$$\begin{bmatrix} 1 & 2 & -1 \\ -4 & -7 & 3 \\ -2 & -6 & 4 \end{bmatrix}$$

Solution:
determinant: 0. This matrix is not invertible

(f)

$$\begin{bmatrix} 1 & 2 & 0 \\ -2 & 4 & 2 \\ 1 & 0 & -1 \end{bmatrix}$$

Solution:
determinant: -4 inverse: $\begin{bmatrix} 1 & -1/2 & -1 \\ 0 & 1/4 & 1/2 \\ 1 & -1/2 & -2 \end{bmatrix}$

(g)

$$\begin{bmatrix} 1 & 0 & -1 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ -1 & 2 & 0 & -2 \end{bmatrix}$$

Solution:
determinant: 1 inverse: $\begin{bmatrix} 2 & 0 & -2 & 1 \\ -4 & 1 & 4 & -2 \\ -9 & 8 & -5 & \\ -5 & 1 & 5 & -3 \end{bmatrix}$

14. Compute the inverse of the following matrix:
$\begin{bmatrix} 2 & 3 \\ 2 & 2 \end{bmatrix}$

Solution: $\begin{bmatrix} -1 & 3/2 \\ 1 & -1 \end{bmatrix}$

15. Use the above inverse to solve the following systems of equations

$$\begin{array}{ccccc} 2x_1 & + & 3x_2 & = & -6 \\ 2x_1 & + & 2x_2 & = & 15 \end{array}$$

Solution: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 57/2 \\ -21 \end{bmatrix}$

16. Compute the determinant of the following matrices (you may use any method you see fit).

(a) $\begin{bmatrix} 5 & \pi \\ -3 & e \end{bmatrix}$

Solution: $5e + 3\pi$

(b) $\begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

Solution: 15

(c) $\begin{bmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{bmatrix}$

Solution: -36

17. Compute the determinant of the following matrices using row/column expansion.

(a) $\begin{bmatrix} 2 & -5 & 4 \\ 0 & 1 & -2 \\ 1 & 0 & 3 \end{bmatrix}$

Solution: 12

(b) $\begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$

Solution: 15

(c) $\begin{bmatrix} 2 & 1 & 0 & 2 \\ 0 & -1 & 0 & 1 \\ -6 & 0 & 2 & 0 \\ 1 & -4 & 0 & 6 \end{bmatrix}$

Solution: -2

18. Complete the following sentences:

(a) A square matrix A is invertible if and only if $\det A$ $\underline{\neq 0}$

(b) If a square matrix A is singular, the columns of A are linearly <u>dependent</u>

(c) If a square matrix A is invertible, the matrix equation $A\overline{x} = \overline{b}$ has <u>a unique</u> solution(s).

19. Suppose A is a square matrix such that $A^4 = 0$. Explain why A cannot be invertible.

Solution: Taking determinants on both sides gives $\det(A^4) = 0$. Now, notice that

$$\det(A^4) = \det(AAAA) = \det(A)\det(A)\det(A)\det(A) = \det(A)^4$$

and therefore $\det(A)^4 = 0$, which is possible only if $\det(A) = 0$. Thus, A is singular, i.e. not invertible.

20. Suppose that A is a square matrix such that $A^4 = I_n$. Explain why A is invertible.

Solution: Again, taking determinants on both sides gives $\det(A^4) = \det(I_n)$. Notice that $\det(I_n) = 1$, and so

$$\det(A)^4 = 1.$$

Thus, $\det(A) = \pm 1$, and in particular is not 0, so A is invertible.

21. Let A be a 5x5 matrix with $\det A = 4$. What is the determinant of

(a) A^2

Solution: 16.

(b) $3A$

Solution: $972 = 4 \cdot 3^5$. Notice that scaling the matrix by 3 scales each row by 3, and since there are 3 rows in the matrix, this changes the determinant by 3 for each row, i.e. 3^5 overall.