Name:

Final Exam - Practice Questions

NOTE: This (mostly) only covers material past the second exam. Please refer to previous practice questions for material from Test 1 and Test 2.
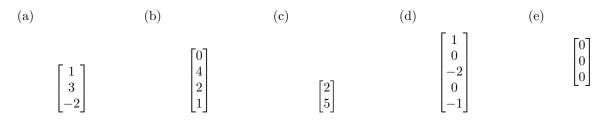
The questions are broken into 3 categories: fundamental questions, advanced questions(*), and challenge questions(**). The fundamental questions test a basic understanding of the definitions and processes. Advanced questions test applying that understanding to more complicated problems. Finally, challenge questions are questions that are intended to stump you. Good luck!

1. Define the following terms:

   - Dot product
   - Inner product
   - Norm (of a vector)
   - Orthogonal vectors
   - Orthogonal set
   - Orthogonal basis

   - Orthogonal projection of $\vec{y}$ onto $\vec{u}$
   - Unit vector
   - Orthonormal set
   - Orthonormal basis
   - Least-squares solution to $A\vec{x} = \vec{b}$
   - Least-squares error

2. Rank the following vectors from greatest to least in terms of their norm:

   (a) $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$
   (b) $\begin{bmatrix} 0 \\ 4 \\ 2 \\ 1 \end{bmatrix}$
   (c) $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$
   (d) $\begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ -1 \end{bmatrix}$
   (e) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

   Solution: (c) ¿ (b) ¿ (a) ¿ (d) ¿ (e)

3. For each of the above vectors, find a unit vector that points in the same direction.

   Solution: (a) $\begin{bmatrix} 1/\sqrt{14} \\ 3/\sqrt{14} \\ -2/\sqrt{14} \end{bmatrix}$, (b) $\begin{bmatrix} 0 \\ 4/\sqrt{21} \\ 2/\sqrt{21} \\ 1/\sqrt{21} \end{bmatrix}$, (c) $\begin{bmatrix} 2/\sqrt{29} \\ 5/\sqrt{29} \end{bmatrix}$, (d) $\begin{bmatrix} 1/\sqrt{6} \\ 0 \\ -2/\sqrt{6} \\ 0 \\ -1/\sqrt{6} \end{bmatrix}$, (e) Does not apply, this vector has no direction.

4. Find a unit vector in $\mathbb{R}^2$ that is orthogonal to $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Solution: We want to find a vector $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ such that $\vec{v} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 0$. Evaluating this dot product gives the equation $-v_1 + 2v_2 = 0$ so

$$v_1 = 2v_2.$$

Thus, any vector of the form $\vec{u} = v_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is orthogonal to $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$. Let us take the vector corresponding to $v_2 = 1$, that is, let $\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Now we would like a unit vector, but this vector has norm $\sqrt{5}$. Dividing by the norm (length) will yield a unit vector, so our answer is: $\vec{v} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$. Note that you could also have $\begin{bmatrix} -2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$ as a solution.

5. Determine which of the following sets are orthogonal sets:

(a)

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} \right\}$$

(b)

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} \right\}$$

(c)

$$\left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix} \right\}$$

Solution: (a) is orthogonal, (b) is not orthogonal, and (c) is orthogonal.

6. Find a non-zero vector $\vec{v}$ in $\mathbb{R}^3$ to make the following set an orthogonal set:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \vec{v} \right\}$$

Is the above set (with your selected $\vec{v}$) a basis for $\mathbb{R}^3$? Why does it HAVE to be a basis?

Solution: One such vector is $\vec{v} = \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$ (use techniques similar to question 9). It has to be a basis since an orthogonal set is also linearly independent, and since the dimension of $\mathbb{R}^3$ is 3, any set of three linearly independent vectors must be a basis for $\mathbb{R}^3$ (think about pivots).

7. Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$. Calculate $\text{proj}_{\vec{v}} \vec{u}$ for the following vectors $\vec{v}$:

(a)
$$\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

(b)
$$\vec{v} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$$

(c)
$$\vec{v} = \begin{bmatrix} 0 \\ -1 \\ 7 \end{bmatrix}$$

8. Let $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$. Calculate $\text{proj}_{\text{Col } A}\, \vec{u}$ for the following matrices A:

(a)
$$A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$$

(b)
$$A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ -1 & -3 \end{bmatrix}$$

9. For $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, find a vector $\vec{v} \neq \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$ so that $\text{proj}_{\vec{u}}\, \vec{v} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$.

10. Find the closest vector to $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ in the subspace

$$W = \text{span}\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

How far is the vector from $\vec{u}$?

11. Use the Gramm-Schmidt process to find an orthogonal basis for the column space of the following matrix:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

12. Find the least-squares solution to the following system of equations:

$$\begin{bmatrix} 3 & -1 \\ 1 & -1 \\ 0 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}$$

What is the least-squares error?

Solution: The least-squares solution is: $\vec{x} = \begin{bmatrix} 33/41, 11/82 \end{bmatrix}$. The least squares error is: $\frac{\sqrt{1697290}}{574}$ (hah) or about 2.26969

13. Find a linear model that best fits the following data points:

$$(1, 5), (2, 4), (4, 1), (5, 1)$$

Solution: $y = 6.05 - 1.1x$

14. Find a quadratic model that best fits the following data points:

$$(1, 5), (2, 1), (4, 1), (5, 4)$$

Solution: $y = \frac{7}{6}x^2 - \frac{36}{5}x + \frac{164}{15}$