Name:

Final Exam - Practice Questions

NOTE: This (mostly) only covers material past the second exam. Please refer to previous practice questions for material from Test 1 and Test 2.

1. Define the following terms:

   - Dot product
   - Inner product
   - Norm (of a vector)
   - Orthogonal vectors
   - Orthogonal set
   - Orthogonal basis
   - Orthogonal projection of $\vec{y}$ onto $\vec{u}$
   - Unit vector
   - Orthonormal set
   - Orthonormal basis
   - Least-squares solution to $A\vec{x} = \vec{b}$
   - Least-squares error

2. Rank the following vectors from greatest to least in terms of their norm:

   (a) $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ (b) $\begin{bmatrix} 0 \\ 4 \\ 2 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ -1 \end{bmatrix}$ (e) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

3. For each of the above vectors, find a unit vector that points in the same direction.

4. Find a unit vector in $\mathbb{R}^2$ that is orthogonal to $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

5. Determine which of the following sets are orthogonal sets:

   (a) $\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -2 \\ 7/2 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} \right\}$ (c) $\left\{ \begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 7 \\ 0 \end{bmatrix} \right\}$

6. Find a non-zero vector $\vec{v}$ in $\mathbb{R}^3$ to make the following set an orthogonal set:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \vec{v} \right\}$$

   Is the above set (with your selected $\vec{v}$) a basis for $\mathbb{R}^3$? Why does it HAVE to be a basis?

7. Let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$. Calculate $\text{proj}_{\vec{v}}\, \vec{u}$ for the following vectors $\vec{v}$:

(a) $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$

(b) $\vec{v} = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$

(c) $\vec{v} = \begin{bmatrix} 0 \\ -1 \\ 7 \end{bmatrix}$

8. Let $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$. Calculate $\text{proj}_{\text{Col}\, A}\, \vec{u}$ for the following matrices A:

(a) $A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \\ -1 & -3 \end{bmatrix}$

9. For $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, find a vector $\vec{v} \neq \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$ so that $\text{proj}_{\vec{u}}\, \vec{v} = \begin{bmatrix} 6 \\ 3 \\ 0 \end{bmatrix}$.

10. Find the closest vector to $\vec{u} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ in the subspace

$$W = \text{span}\left\{ \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

How far is the vector from $\vec{u}$?

11. Use the Gramm-Schmidt process to find an orthogonal basis for the column space of the following matrix:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

12. Find the least-squares solution to the following system of equations:

$$\begin{bmatrix} 3 & -1 \\ 1 & -1 \\ 0 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 3 \end{bmatrix}$$

What is the least-squares error?

13. Find a linear model that best fits the following data points:

$$(1, 5), (2, 4), (4, 1), (5, 1)$$

14. Find a quadratic model that best fits the following data points:

$$(1, 5), (2, 1), (4, 1), (5, 4)$$